Prescribing Gaussian curvature on compact surfaces and geodesic curvature on its boundary

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Outline

Prescribing Gaussian curvature under conformal changes of the metric

A classical problem in geometry is the prescription of the Gaussian curvature on a compact Riemannian surface Σ under a conformal change of the metric.

Denote by $\tilde{g}$ the original metric and $g = e^u \tilde{g}$. The curvature then transforms according to the law:

$$-\Delta u + 2\tilde{K}(x) = 2K(x)e^u,$$

where $\Delta = \Delta_{\tilde{g}}$ is the Laplace-Beltrami operator and $\tilde{K}$, K stand for the Gaussian curvatures with respect to $\tilde{g}$ and g, respectively.

The solvability of this equation has been studied for several decades: Berger, Kazdan and Warner, Moser, Aubin, Chang-Yang...

Our problem

Let Σ be a compact surface with boundary. In this talk we consider the problem of prescribing the Gaussian curvature of Σ and the geodesic curvature of $\partial\Sigma$ via a conformal change of the metric.

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This question leads us to the boundary value problem:

$$\begin{cases} -\Delta u + 2\tilde{K}(x) = 2K(x)e^{u}, & x \in \Sigma, \\ \dfrac{\partial u}{\partial \nu} + 2\tilde{h}(x) = 2h(x)e^{u/2}, & x \in \partial\Sigma. \end{cases}$$

Here e^u is the conformal factor, ν is the normal exterior vector and

1. $\tilde{K}$, $\tilde{h}$ are the original Gaussian and geodesic curvatures, and
2. K, h are the Gaussian and geodesic curvatures to be prescribed.

Antecedents

- The higher order analogue: prescribing scalar curvature S on Σ and mean curvature H on $\partial\Sigma$.

  The case $S = 0$ and $H = const$ is the well-known Escobar problem: Ambrosetti-Li-Malchiodi, Escobar, Han-Li, Marques,...

- The case $h = 0$: Chang-Yang.

- The case $K = 0$: Chang-Liu, Liu-Huang...

  The blow-up phenomenon has also been studied: Guo-Liu, Bao-Wang-Zhou, Da Lio-Martinazzi-Rivière...

- The case of constants K, h:

  A parabolic flow converges to constant curvatures (Brendle).

  Classification of solutions in the annulus (Jiménez).

  Classification of solutions in the half-plane (Li-Zhu, Zhang, Gálvez-Mira).

Our aim is to consider the case of nonconstant K, h. The only results we are aware of are due to Cherrier, Hamza.

Preliminaries

By the Gauss-Bonnet Theorem,

$$\int_\Sigma K e^u + \oint_{\partial\Sigma} h e^{u/2} = \int_\Sigma \tilde{K} + \oint_{\partial\Sigma} \tilde{h} = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ.

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where $\chi(\Sigma)$ is the Euler characteristic of Σ.

It is easy to show that we can prescribe $h = 0$, $K = sgn(\chi(\Sigma))$. Then:

$$\begin{cases} -\Delta u + 2\tilde{K} = 2K(x)e^u, & x \in \Sigma, \\ \dfrac{\partial u}{\partial \nu} = 2h(x)e^{u/2}, & x \in \partial\Sigma, \end{cases}$$

where $\tilde{K} = sgn(\chi(\Sigma))$.

We are interested in the case of negative K. For existence of solutions, we will focus on the case $\chi \leq 0$.

The variational formulation

The associated energy functional is given by $I : H^1(\Sigma) \to \mathbb{R}$,

$$I(u) = \int_\Sigma \left(\frac{1}{2} |\nabla u|^2 + 2\tilde{K} u + 2|K(x)| e^u \right) - 4 \oint_{\partial\Sigma} h e^{u/2}.$$

For the statement of our results it will be convenient to define the function $\mathfrak{D} : \partial\Sigma \to \mathbb{R}$,

$$\mathfrak{D}(x) = \frac{h(x)}{\sqrt{|K(x)|}}.$$

The function $\mathfrak{D}$ is scale invariant.

A trace inequality

Proposition

For any $\varepsilon > 0$ *there exists* $C > 0$ *such that:*

$$4\int_{\partial\Sigma} h(x)e^{u/2} \leq (\varepsilon + \max_{p\in\partial\Sigma} \mathfrak{D}^+(p)) \left[\int_\Sigma \frac{1}{2}|\nabla u|^2 + 2|K(x)|e^u\right] + C.$$

In particular, if $\mathfrak{D}(p) < 1 \; \forall \, p \in \partial\Sigma$, then I is bounded from below.

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Assume that $h > 0$ is constant, and take N a vector field in Σ such that $N(x) = \nu(x)$ on the boundary, $|N(x)| \leq 1$. Then,

$$4 \int_{\partial \Sigma} h e^{u/2} = 4 \int_{\partial \Sigma} h e^{u/2} N(x) \cdot \nu(x)$$

$$= 4 \int_\Sigma h e^{u/2} \left[div \, N + \frac{1}{2} \nabla u \cdot N \right] \leq C \int_\Sigma e^{u/2} + 2 \int_\Sigma h e^{u/2} |\nabla u|$$

$$\leq C \int_\Sigma e^{u/2} + 2 \int_\Sigma h^2 e^u + \frac{1}{2} \int_\Sigma |\nabla u|^2.$$

The case $\chi(\Sigma) < 0$

Theorem

Assume that $\chi(\Sigma) < 0$. Let K, h be continuous functions such that $K < 0$ and $\mathfrak{D}(p) < 1$ for all $p \in \partial\Sigma$. Then the functional I is coercive and attains its infimum.

By the trace inequality,

$$I(u) \geq \int_\Sigma \varepsilon|\nabla u|^2 + 2\varepsilon|K(x)|e^u + 2\tilde{K}u - C.$$

Since $\tilde{K} < 0$, $\lim_{u\to\pm\infty} 2\delta e^u + 2\tilde{K}u = +\infty$, so I is coercive.

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If $\chi(\Sigma) = \tilde{K} = 0$, I is bounded from below but not coercive!
The reason is that $\int_\Sigma u_n$ could go to $-\infty$ for a minimizing sequence u_n.

Minimizers for $\chi(\Sigma) = 0$.

Theorem

Assume that $\chi(\Sigma) = 0$. Let K, h be continuous functions such that $K < 0$ and:

1. $\mathfrak{D}(p) < 1$ *for all* $p \in \partial\Sigma$.
2. $\oint_{\partial\Sigma} h > 0$.

Then I attains its infimum.

Observe that if $u_n = -n$, then: $I(u_n) = \int_\Sigma 2|K(x)|e^{-n} - 4 \oint_{\partial\Sigma} he^{-n/2} \nearrow 0$.

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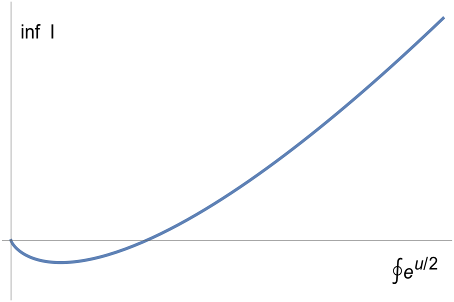


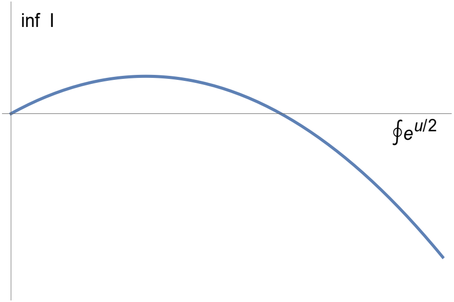
Min-max for $\chi(\Sigma) = 0$.

Theorem

Assume that $\chi(\Sigma) = 0$. Let K, h be continuous functions such that $K < 0$ and:

1. $\mathfrak{D}(p) > 1$ *for some* $p \in \partial\Sigma$.
2. $\oint_{\partial\Sigma} h < 0$.

Then I has a mountain-pass geometry.



Blow-up versus compactness

Here the (PS) condition is not known to hold. By using the monotonicity trick of Struwe, we can obtain solutions of perturbed problems.

The question of compactness or blow-up for this kind of problems has attracted a lot of attention since the works of Brezis-Merle, Li-Shafrir, etc.

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Let u_n be a blowing-up sequence (namely, $\sup\{u_n(x)\} \to +\infty$) of solutions to the problem:

$$\begin{cases} -\Delta u_n + 2\tilde{K}_n(x) = 2K_n(x)e^{u_n}, & \text{in } \Sigma, \\ \dfrac{\partial u_n}{\partial \nu} + 2\tilde{h}_n(x) = 2h_n(x)e^{u_n/2}, & \text{on } \partial\Sigma. \end{cases} \tag{1}$$

Here $\tilde{K}_n \to \tilde{K}$, $\tilde{h}_n \to \tilde{h}$, $K_n \to K$, $h_n \to h$ in C^1 sense, with $K < 0$. By integrating:

$$\int_\Sigma K_n e^{u_n} + \oint_{\partial\Sigma} h_n e^{u_n/2} = \int_\Sigma K_n + \oint_{\partial\Sigma} h_n \to \chi_0 = 2\pi\chi(\Sigma).$$

Hence there could be compensation of diverging masses!!

A blow-up analysis

Theorem

Assume that u_n is unbounded from above and define its singular set:

$$S = \{p \in \Sigma : \ \exists\, x_n \to p \text{ such that } u_n(x_n) \to +\infty\}. \tag{2}$$

1. $S \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1\}$.

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1. $S \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1\}$.
2. *If $\int_\Sigma e^{u_n}$ is bounded, then there exists $m \in \mathbb{N}$ such that*

$$S = \{p_1, \dots p_m\} \subset \{\mathfrak{D}(p) > 1\}.$$

In this case $|K_n| e^{u_n} \rightharpoonup \sum_{i=1}^m \beta_i \delta_{p_i}$, $h_n e^{u_n/2} \rightharpoonup \sum_{i=1}^m (\beta_i + 2\pi)\delta_{p_i}$ for some $\beta_i > 0$. In particular, $\chi_0 = 2\pi m$.

The infinite mass case

3. If $\int_\Sigma e^{u_n}$ is unbounded, there exists a unit positive measure σ on Σ such that:

   a) $\dfrac{|K_n| e^{u_n}}{\int_\Sigma |K_n| e^{u_n}} \rightharpoonup \sigma, \qquad \dfrac{h_n e^{u_n/2}}{\oint_{\partial\Sigma} h_n e^{u_n/2}} \rightharpoonup \sigma|_{\partial\Sigma}.$

   b) $supp\ \sigma \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1, \ \mathfrak{D}_\tau(p) = 0\}.$

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   b) $supp\ \sigma \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) \geq 1, \ \mathfrak{D}_\tau(p) = 0\}.$

4. If there exists $m \in \mathbb{N}$ such that $ind(u_n) \leq m$ for all n, then $S = S_0 \cup S_1$, where:

$$S_0 \subset \{p \in \partial\Sigma : \ \mathfrak{D}(p) = 1, \ \mathfrak{D}_\tau(p) = 0\},$$

$$S_1 = \{p_1, \dots p_k\} \subset \{\mathfrak{D}(p) > 1 \text{ and } \Phi(p) = 0\}, \quad k \leq m.$$

   If moreover $\chi_0 \leq 0$, then S_1 is empty.

Back to the case $\chi(\Sigma) = 0$.

Theorem

Assume that $\chi(\Sigma) = 0$. Let K, h be C^1 functions such that $K < 0$ and:

1. $\mathfrak{D}(p) > 1$ *for some* $p \in \partial\Sigma$.
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1. $\mathfrak{D}(p) > 1$ *for some* $p \in \partial\Sigma$.
2. $\oint_{\partial\Sigma} h < 0$.
3. $\mathfrak{D}_\tau(p) \neq 0$ *for any* $p \in \partial\Sigma$ *with* $\mathfrak{D}(p) = 1$.

Then I has a mountain-pass critical point.

We obtain solutions of perturbed problems of mountain-pass type, hence they have Morse index at most 1 ([Fang-Ghoussoub, 94, 99]).

Those solutions cannot blow-up so that they converge to a true solution of our problem.

Obstructions to existence

Proposition (Jiménez 2012)

If Σ is an cylinder and $K = -1$, h_1 and h_2 are constants, then our problem is solvable iff

1. $h_1 + h_2 > 0$ *and both* $h_i < 1$ *(minima).*
2. $h_1 + h_2 < 0$ *and some* $h_i > 1$ *(mountain-pass).*
3. $h_1 = 1$, $h_2 = -1$ *or viceversa.*

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Proposition

Let Σ be a compact surface with boundary, and assume that $h(p) > \sqrt{|K^-(q)|}$ for all $p \in \partial\Sigma$, $q \in \Sigma$. Then Σ is homeomorphic to a disk.

A classification result in the half-plane

Theorem (Gálvez-Mira 2009)

Let u be a solution of:

$$\begin{cases} -\Delta u = 2K_0 e^u & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} = 2h_0 e^{u/2} & \text{in } \partial\mathbb{R}^2_+, \end{cases} \implies \begin{cases} -\Delta u = -2e^u & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} = 2\mathfrak{D}_0 e^{u/2} & \text{in } \partial\mathbb{R}^2_+. \end{cases}$$

with $\mathfrak{D}_0 = \dfrac{h_0}{\sqrt{|K_0|}}$. Then the following holds:

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with $\mathfrak{D}_0 = \dfrac{h_0}{\sqrt{|K_0|}}$. Then the following holds:

- *If $\mathfrak{D}_0 < 1$ there is no solution.*
- *If $\mathfrak{D}_0 = 1$ the only solutions are:*

$$u(s,t) = 2\log\left(\frac{\lambda}{1+\lambda t}\right), \ \lambda > 0, \ s \in \mathbb{R}, \ t \geq 0.$$

A classification result in the half-plane

- *If* $\mathfrak{D}_0 > 1$*, then:*

$$u(z) = 2\log\left(\frac{2|g'(z)|}{1-|g(z)|^2},\right),$$

where g *is locally injective holomorphic map from* $\mathbb{R}^2_+$ *to a disk of geodesic curvature* $\mathfrak{D}_0$ *in the Poincaré disk* $\mathbb{H}^2$*. For instance, to* $B(0,R)$ *with* $\mathfrak{D}_0 = \frac{1+R^2}{2R}$*.*

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Moreover, g *is a Möbius transform if and only if*

$$\text{either } \int_{\mathbb{R}^2_+} e^u < +\infty \text{ and / or } \oint_{\partial\mathbb{R}^2_+} e^{u/2} < +\infty.$$

In such case u *can be written as:*

$$u(s,t) = 2\log\left(\frac{2\lambda}{(s-s_0)^2+(t+t_0)^2-\lambda^2}\right), t \geq 0,$$

where $\lambda > 0$, $s_0 \in \mathbb{R}$, $t_0 = \mathfrak{D}_0\lambda$*. We call these solutions "bubbles".*

Passing to a limit problem in the half-plane

Let us recall the definition of the singular set:

$$S = \{p \in \Sigma : \exists y_n \in \Sigma, \ y_n \to p, \ u_n(y_n) \to +\infty\}.$$

Proposition

Let $p \in S$. Then there exist $x_n \in \Sigma$, $x_n \to p$ such that, after a suitable rescaling, we obtain a solution of the problem in the half-plane in the limit.

In particular $S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1\}$.

In the Lioville equation, if the mass is finite, then a key integral estimate ([Brezis-Merle, 1991]) implies that S is finite. Hence one can take x_n as local maxima ([Li-Shafrir, 1994]).

Here the idea is to choose a **good sequence** x_n, even if they are not local maxima!

Choosing good sequences

Let us fix $p \in S$. Via a conformal map we can pass to either $B_0(r)$ or $B_0^+(r)$.

By definition there exist $y_n \in \Sigma$ with $y_n \to p$ and $u_n(y_n) \to +\infty$. Define:

$$\varphi_n = e^{-\frac{u_n}{2}}, \quad \varepsilon_n = e^{-\frac{u_n(y_n)}{2}} \to 0.$$

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By Ekeland variational principle there exists a sequence x_n such that

- $e^{-\frac{u_n(x_n)}{2}} \leq e^{-\frac{u_n(y_n)}{2}}$,
- $|x_n - y_n| \leq \sqrt{\varepsilon_n}$,
- $e^{-\frac{u_n(x_n)}{2}} \leq e^{-\frac{u_n(z)}{2}} + \sqrt{\varepsilon_n}\,|x_n - z|$ for every $z \in B$.

The last conditions implies that, when we rescale, the rescaled functions are bounded from above, so we can pass to a limit.

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The last conditions implies that, when we rescale, the rescaled functions are bounded from above, so we can pass to a limit.

- Since $K(p) < 0$, there is no entire solution of $-\Delta u = 2K(p)e^u$ in $\mathbb{R}^2$.
- Hence p in $\partial\Sigma$, the limit problem is posed in a half-plane and $\mathfrak{D}(p) \geq 1$.

Infinite mass

Proposition

Assume that $\rho_n = \int_\Sigma |K_n| e^{u_n} \to +\infty$, $\oint_{\partial\Sigma} |h| e^{u_n/2} \to +\infty$. *Then there exists a positive unit measure* σ *on* $\partial\Sigma$ *such that:*

$$\frac{|K_n|e^{u_n}}{\rho_n} \rightharpoonup \sigma, \quad \frac{h_n e^{u_n/2}}{\rho_n} \rightharpoonup \sigma.$$

Multiplying the equation by $\phi \in C^2(\Sigma)$ and integrating:

$$2\oint_{\partial\Sigma} h_n e^{u_n/2}\phi - 2\int_\Sigma |K_n| e^{u_n}\phi = O(1) + \underbrace{\int_\Sigma u_n \Delta\phi + \oint_{\partial\Sigma} \frac{\partial\phi}{\partial\nu} u_n}_{o(\rho_n)}.$$

We use a Kato-type inequality to estimate u_n^-.

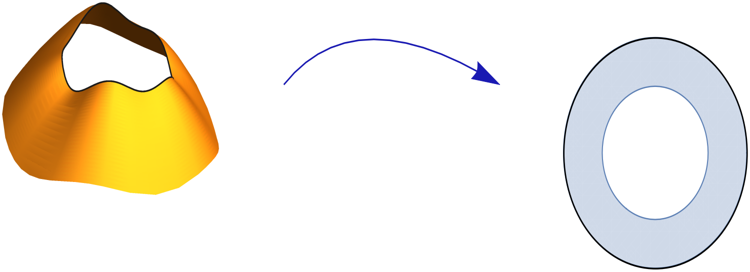
On the support of σ

Clearly $supp\ \sigma \subset S \subset \{p \in \partial\Sigma : \mathfrak{D}(p) \geq 1\}$. Moreover, we have:

Proposition

The support of σ is contained in the set $\{p \in \partial\Sigma : \mathfrak{D}_\tau(p) = 0\}$.

Let Λ_0 be a connected component of $\partial\Sigma$. Via a conformal map, we can pass to a problem in an annulus.



Multiply the equation by $\nabla u_n \cdot F$, where F is a tangential vector field, to obtain:

$$\oint_{\Lambda_0} (4h_n e^{u_n/2} - 4\tilde{h}_n)(\nabla u_n \cdot F)$$

$$= \int_\Sigma [4\tilde{K}_n \nabla u_n \cdot F + 4e^{u_n}(\nabla K_n \cdot F + K_n \, \nabla \cdot F) + 2\underbrace{DF(\nabla u_n, \nabla u_n) - \nabla \cdot F|\nabla u_n|^2}_{??}].$$

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We get rid of the Dirichlet term by using holomorphic functions F. Integrating by parts and passing to the limit, we obtain:

$$\oint_{\Lambda_0} \frac{\mathfrak{D}_\tau}{\mathfrak{D}} f \, d\sigma = 0,$$

where $f = (F \cdot \tau)$. But f can be any arbitrary analytic function, and then $\mathfrak{D}_\tau \sigma = 0$ as a measure.

Morse index

This is all the information that we can obtain without further assumptions on u_n.

From now on we assume that the sequence of solutions u_n has bounded Morse index.

If u_n has bounded Morse index, the solutions of the limit problem obtained by rescaling have finite Morse index.

Morse index of the limit problem

Theorem

Let u be a solution of the problem:

$$\begin{cases} -\Delta u = -2e^{u} & \text{in } \mathbb{R}^2_+, \\ \dfrac{\partial u}{\partial \nu} = 2\mathfrak{D}_0 e^{u/2} & \text{in } \partial\mathbb{R}^2_+. \end{cases} \tag{3}$$

Define:

$$Q(\psi) = \int_{\mathbb{R}^2_+} |\nabla \psi|^2 + 2\int_{\mathbb{R}^2_+} e^{u}\psi^2 - \mathfrak{D}_0 \int_{\partial\mathbb{R}^2_+} e^{u/2}\psi^2, \text{ and}$$

$$ind(v) = \sup\{dim(E) : \; E \subset C_0^\infty(\mathbb{R}^2_+) \text{ vector space}, \, Q(\psi) < 0 \, \forall \, \psi \in E\}.$$

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$$ind(v) = \sup\{dim(E) : E \subset C_0^\infty(\mathbb{R}^2_+) \text{ vector space}, Q(\psi) < 0 \ \forall \psi \in E\}.$$

1. *If $\mathfrak{D}_0 = 1$, then $ind(u) = 0$, that is, u is stable.*
2. *If $\mathfrak{D}_0 > 1$ and u is a bubble, then $ind(u) = 1$. Otherwise, $ind(u) = +\infty$.*

This theorem implies that infinite mass blow-up with bounded Morse index occurs only if $\mathfrak{D}(p) = 1$, and the number of bubbles is limited.

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- For a convenient cut-off ϕ, $\psi = \phi(g \circ \gamma)$ satisfies $Q(\psi) < 0$.
- If moreover $\oint_{\partial\mathbb{R}^2_+} e^{u/2} = +\infty$ we can choose ϕ to be 0 outside any arbitrary compact set.

Explicit examples of blow-up

Let us consider the problem:

$$\begin{cases} -\Delta u = -2e^{u}, & \text{in } A(0;r,1), \\ \frac{\partial u}{\partial \nu} + 2 = 2h_1 e^{u/2}, & \text{on } |x| = 1, \\ \frac{\partial u}{\partial \nu} - \frac{2}{r} = 2h_2 e^{u/2}, & \text{on } |x| = r. \end{cases}$$

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For example, the function:

$$u(x) = \log\left(\frac{4}{|x|^2(\lambda + 2\log|x|)^2}\right), \quad \text{for any } \lambda < 0,$$

is a solution with $h_1 = 1$ and $h_2 = -1$. Observe that if λ tends to 0 then u blows up at a whole component of the boundary.

The singular set $S = \{|x| = 1\}$ is not finite.

A second example

Given any $h_1 > 1$, $\gamma \in \mathbb{N}$, there exists a explicit solution:

$$u_\gamma(z) = 2\log\left(\frac{\gamma|z|^{\gamma-1}}{h_1 + Re(z^\gamma)}\right),$$

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The asymptotic profile is:

$$u(s,t) = 2\log\left(\frac{e^{-t}}{h_1 + e^{-t}\cos s}\right),$$

defined in the half-plane $\{t \geq 0\}$. This is indeed a solution to the limit problem in the half-space with $K = -1$ and $h_1 > 1$, with infinite Morse index.

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Muito obrigado pela sua atenção!